Mathematics for Classical Dynamics

Spatial dynamics can be conveniently analyzed in a six-dimensional formalism. Those of you who are familiar with the kinematic concepts of screws, motors, duals, binors, or dyadics may find it useful to compare those formalisms with this one.

Any spatial vector is composed of a free part and a line part. The free part is like a torque vector, a wind, or an angular velocity – it contains just a magnitude and direction. The line part is like a force vector, a bullet trajectory, or a displacement – it contains magnitude and direction, but it also travels along a particular line in space. Any spatial vector

$$\hat{a} = \begin{bmatrix} \alpha \\ a_O \end{bmatrix}$$

is a line vector if

$$\alpha \cdot a_O = 0, \text{ where } a_O = \vec{A} \times \alpha$$

because it represents the vector α passing through the point $\vec{A}$. Canonically, if the first component of a spatial vector is zero, it is a free vector, whereas if the second is zero, it is a line vector.

A rigid body's velocity is completely specified by its angular velocity and the velocity of one point on the rigid body. For convenience, we agree that a spatial velocity

$$\hat{v} = \begin{bmatrix} \omega \\ v_O \end{bmatrix}$$

is a composite vector consisting of the angular velocity ω and the velocity of the point instantaneously at the origin, v_O. Thus the linear velocity of a point P on the body is $v_P = v_O + \vec{P} \times \omega$, by the "shifting rule" of classical mechanics.

The formula for a rigid-body coordinate transform can be found by considering the translation and rotation cases separately. Since translating the origin from O to P changes

$$\hat{a} = \begin{bmatrix} \alpha \\ a_O \end{bmatrix} \text{ to } \begin{bmatrix} \alpha \\ a_O + \vec{P} \times \alpha \end{bmatrix}$$

and a rotation changes the line and free component vectors by cartesian rotation, the transformation which shifts O to P followed by rotation by E about P is expressed by

$${}_P\hat{X}_O = \begin{bmatrix} E & 0 \\ Er\times^T & E \end{bmatrix}$$

where $r\times$ is the operator which crosses r by a vector. Its inverse is

$${}_O\hat{X}_P = \begin{bmatrix} E^{-1} & 0 \\ r \times E^{-1} & E^{-1} \end{bmatrix}$$

so right away we see that transformations are different in spatial coordinates.

Next, we consider differentiation in moving coordinates. Let point P be moving at velocity $\hat{v}_O$. Then $\frac{\partial}{\partial t} {}_O\hat{X}_P = \hat{v}_O \hat{\times}_O \hat{X}_P$, where $\frac{\partial}{\partial t}$ denotes component-wise differentiation and $\hat{\times}$ is the spatial cross operator, defined so that $\begin{bmatrix} a \\ b \end{bmatrix} \hat{\times} = \begin{bmatrix} a\times & 0 \\ b\times & a\times \end{bmatrix}$. This can be derived by transforming a vector into a stationary coordinate system, taking the derivative, and transforming it back. Finally, we conclude that if $\hat{a}$ is moving with velocity $\hat{v}$, then the derivative of $\hat{a}$ is $\hat{v} \hat{\times} \hat{a}$.

Next we arrive at the concept of spatial acceleration. Since spatial acceleration has two components consisting of the derivative of the angular acceleration and the derivative of the linear velocity of the point instantaneously at the origin, this means that centripetal acceleration is zero. If the earth revolves in a perfect circle around the sun, then it has *zero* centripetal acceleration.

Next, we consider joint variables. Consider an *n* degree-of-freedom joint *j*. Then if q_i is the value of the i^{th} joint variable, and $\hat{S}$ is the "joint subspace matrix" (that, is the, matrix such that the velocity and acceleration of a joint can be specified as

$$\hat{v} = \hat{S}\dot{q} + \hat{v}_{parent},\ \hat{a} = \hat{S}\ddot{q} + \hat{v} \hat{\times} \hat{S}\dot{q} + \hat{a}_{parent} .$$

For example, for a 1-DOF joint which revolves about the z-axis, we would have

$$\hat{S} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

since this extracts the the z-angular component of the angular velocity. This lets you "project out" the active components of a force and add the relevant ones to the current joint's velocity.

On to the physics! We must first find the spatial inertia of a body. We assume that the 3x3 rotational inertia matrix I is known (the diagonal elements are the moments about the x, y, and z axes, respectively, and the other elements are cross- inertias), and that the center of mass is at *P*. We define the spatial inertia $\hat{I}$ to be the tensor which relates spatial velocity with spatial momentum. If

$$\begin{aligned} \hat{P} &= \text{linear momentum } + \text{ angular momentum} \\ &= \begin{bmatrix} mv_P \\ P\times mv_p \end{bmatrix} + \begin{bmatrix} 0 \\ I\omega \end{bmatrix} \\ &= \begin{bmatrix} m(v_O + (-P)\times\omega) \\ I\omega + P\times m(v_O + (-P)\times\omega) \end{bmatrix} \\ &= \begin{bmatrix} m(-P)\times & m1 \\ I + P\times\ m\,(-P)\times & P\times\ m \end{bmatrix} \hat{v} \end{aligned}$$

and so we have found $\hat{I}$.

Inertias require special treatment. Transforming them from one coordinate frame to another requires conjugation – i.e., $\hat{I}_P = {}_P\hat{X}_O \hat{I}_{O\,O}\hat{X}_P$, which can be verified directly by considering the momentum in the two different frames. The derivative of a spatial inertia is just the component-wise derivative. In particular, if an inertia $\hat{I}$ is moving with velocity $\hat{v}$, then the derivative of $\hat{I}$ is given by $\frac{\partial}{\partial t}\hat{I} = \hat{v}\hat{\times}\hat{I} - \hat{I}\hat{v}\hat{\times}$, as can be verified by directly taking the derivative in two different frames.

Now for forces! Newton's law applies, that is, $\hat{f} = \frac{\partial}{\partial t}\hat{I}\hat{v} = \hat{I}\hat{a} + \hat{v}\hat{\times}\hat{I}\hat{v}$. We may consider this to be an inhomogeneous equation $\hat{f} = \hat{I}\hat{a} + \hat{p}$, where $\hat{p}$ is the "bias force" necessary to give the body zero spatial acceleration.

Finally we consider an articulated-body inertia. The relationship between a force applied to a certain rigid body and its acceleration, taking into account all the aspects of the articulated body, is linear, with the ABI being the tensor. Obviously the ABI depends on the configuration of the links, since as the geometry changes, the response of the rest of the links changes. Fortunately, an ABI is symmetric and positive-definite. The example given in Featherstone is instructive: consider a ball (mass m_1) in a pipe (mass m_2). A force along the axis of the pipe – let's call it the z-axis – will accelerate the ball only, whereas a force along either of the other axes will accelerate both the ball and the pipe. Therefore, the mass portion (upper right corner) of the ABI, from the point of view of the pipe, is

$$\begin{bmatrix} m_1 + m_2 & 0 & 0 \\ 0 & m_1 + m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix}.$$

Now for work. Work is force times displacement. But the scalar product is not quite the same for spatial vectors as it is for ordinary vectors: transposing switches the two sets of three components – it is no longer positive definite, and is only defined between a force-type vector and a displacement-type vector. In this fashion, the line vector of force (linear force) is multiplied times the free vector of displacement (linear displacement) and the free vector of force (angular torque) is multiplied times the line vector of displacement (differential angle). Thus the work w is

$$\hat{f} = \begin{bmatrix} f \\ f_O \end{bmatrix}, \hat{d} = \begin{bmatrix} \partial \\ \partial_O \end{bmatrix}, w = f\partial_O + f_O\partial$$

The spatial transpose is defined by swapping the two components; for a matrix, you simply swap the upper-left and lower-right matrices and take traditional transposes of the four 3x3 matrices. The spatial transpose is designated by a superscript S. The scalar product of two vectors is equal to the spatial tranpose of the first times the second.

"A rigid body inertia . . . can be represented by the sum of six symmetric dyads." This sentence is the heading of a long section of Featherstone, but it's not really necessary to understand in depth – just think of it as splitting an RBI into components over a six-dimensional basis.

Inverse dynamics

Inverse dynamics is a simple problem: given the acclerations, calculate the necessary joint forces. The method we describe is called the Newton- Euler method.

1) Compute all the velocities and accelerations.

$$\hat{v}_i = \hat{S}_i \dot{q}_i + \hat{v}_{i-1} \ (\hat{v}_0 = 0)$$

$$\hat{a}_i = \hat{S}_i \ddot{q}_i + \hat{v}_i \hat{\times} \hat{S}_i \dot{q}_i + \hat{a}_{i-1} \ (\hat{a}_0 = 0)$$

2) Calculate the net force on each link.

$$\hat{f}_i^{net} = \hat{I}_i \hat{a}_i + \hat{v}_i \hat{\times} \hat{I}_i \hat{v}_i$$

3) Calculate the joint forces.

$$\hat{f}_i = \hat{f}_{i+1} + \hat{f}_i^{net} - \hat{f}_i^{ext} \ (\hat{f}_{end} = \hat{f}_{end}^{net})$$

$$Q_i = \hat{S}_i^S \hat{f}_i$$

Forward dynamics

Forward dynamics is harder: given the forces, how will the robot move? There are lots of force-projections and lots of interactions. The articulated-body inertias make it even more complicated, but essentially you treat an n-joint moving robot as a 1-joint robot whose outer link is a vast articulated body.

To solve a 1-joint robot:

Let the base be moving with velocity $\hat{v}_b$ and acceleration $\hat{a}_b$. Let the link have inertia $\hat{I}$, and let the joint have subspace $\hat{S}$. Calculate the velocity, acceleration, and net force of the link as described above. Project out $Q = \hat{S}^S \hat{f}$, then substitute the net force for $\hat{f}$ and the values of the acceleration and velocity into $\hat{f}$. Then solve for $\ddot{q}$.

To calculate articulated-body inertias:

Let an articulated body comprise two members, *1* and *2*. Then a force $\hat{f}$ acting on *1* will be distributed between the two, by Newton's law, according to $\hat{f}_1 = \hat{I}_1 \hat{a}_1,\ \hat{f}_2 = \hat{I}_2 \hat{a}_2$. Furthermore, the joint subspace matrix projects out the component that acts on *2*, since it must be transmitted through the joint. This means that $\hat{a}_2 = \hat{a}_1 + \hat{S}\ddot{q}$ (where here, $\ddot{q}$ is sort of the homogeneous case of the joint velocity), and furthermore since $\hat{f}_2$ does no work in the direction of the joint, $\hat{S}^S \hat{f}_2 = 0$. We can solve these last three equations, getting $\ddot{q}$, and then getting $\hat{f}$ in terms of $\hat{a}_1$. It follows that since $\hat{f} = \hat{I}^A \hat{a}_1$, we have found $\hat{I}^A$. Articulated body inertias do not depend on the forces in the system..

The Final Algorithm: after adding in allowances for nonzero velocities and active forces, the algorithm takes the following form (which is how I implemented it, with several temporary variables to speed up computation):

1) Percolate down from the root of the tree, calculating velocities. In all of these equations, lambda represents parent and nu represents child.

$$\hat{v}_i = {}_i\hat{X}_{\lambda_i} \hat{v}_{\lambda_i} + \hat{S}_i \dot{q}_i \ (\hat{v}_0 = \dot{q}_0)$$

$$\hat{c}_i = \hat{v}_i \hat{\times} \hat{S}_i \dot{q}_i + \dot{\hat{S}}_i \dot{q}_i$$

2) Percolate back up (in some order so that information is never lacking – backwards along the list is fine), calculating ABIs.

$$\hat{I}_i^A = \hat{I}_i + \sum_{v_i} {}_i\hat{X}_{v_i} (\hat{I}_{v_i}^A - \hat{h}_{v_i} \tilde{b}_{v_i} \hat{h}_{v_i}^S) {}_{v_i}\hat{X}_i \quad (\hat{I}_{end}^A = \hat{I}_{end})$$

$$\tilde{b}_i = (\hat{S}_i^S \hat{I}_i^A \hat{S}_i)^{-1}$$

$$\hat{p}_i = (\hat{v}_i \hat{\times} \hat{I}_i \hat{v}_i - \hat{f}_i^{ext}) + \sum_{v_i} {}_i\hat{X}_{v_i} (\hat{p}_{v_i} + \hat{I}_{v_i}^A \hat{c}_{v_i} + \hat{h}_{v_i} \tilde{b}_{v_i} \tilde{u}_{v_i})$$

$$\tilde{u}_i = Q_i - \hat{h}_i^S \hat{c}_i - \hat{S}_i^S \hat{p}_i$$

3) Percolate back down, calculating accelerations.

$$\ddot{q}_i = \tilde{b}_i (\tilde{u}_i - \hat{h}_i {}_i\hat{X}_{\lambda_i} \hat{a}_{\lambda_i})$$

$$\hat{a}_i = {}_i\hat{X}_{\lambda_i} \hat{a}_{\lambda_i} + \hat{c}_i + \hat{S}_i \ddot{q}_i$$

If the root is fixed, then the acceleration of the base is zero; if the root is free, then the acceleration of the root is given by

$$\hat{a}_0 = (\hat{I}_0^A)^{-1} (\hat{f}_0^{ext} - \hat{p}_0), \quad \hat{a}_0 = \ddot{q}_0$$

This concludes the mathematics. More information concerning collisions, contact, composite-rigid-body, hybrid, and other algorithms are in Featherstone's book.

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