Tychonoff theorem: an arbitrary product of compact spaces is compact in the product topology.

Proof: ·1· X a set, $\mathcal{A}$ a collection of subsets of X satisfying f.i.c., then $\exists\, \mathcal{D} \supset \mathcal{A}$ s.t. $\mathcal{D}$ satisfies the f.i.c., but if $\mathcal{E} \supset \mathcal{D}$, $\mathcal{E}$ does not satisfy the f.i.c. "$\mathcal{D}$ is maximal w.r.t. the f.i.c."

we apply a special case of the maximum principle, where B is a 1-elt. subset of A, and apply it to a sSet (a set whose elements are collections of sets). Suppose $\mathcal{A}$ a collection of subsets of X, satisfying the f.i.c., and let $\mathbb{A}$ be the sSet containing all collections of subsets of X that satisfy the f.i.c, with a strict partial order of proper inclusion. Then there is a maximal sSet $\mathbb{C} \subset \mathbb{A}$ that is simply ordered, $\mathcal{A} \in \mathbb{C}$. Let $\mathcal{D} = \bigcup_{\mathcal{C} \in \mathbb{C}} \mathcal{C}$: you can verify that this satisfies the axioms for being maximal w.r.t. the f.i.c.

·2· any finite intersection of elements of $\mathcal{D}$ belongs to $\mathcal{D}$. If B is the intersection of finitely many elements of $\mathcal{D}$, then $\mathcal{E}$ must equal $\mathcal{D}$, where $\mathcal{E} = \mathcal{D} \cup \{B\}$, for else we violate maximality. We just need to show $\mathcal{E}$ satisfies the f.i.c, which is trivial.
if $A \subset X$, A intersects every element of $\mathcal{D}$, $A \in \mathcal{D}$
let $\mathcal{E} = \mathcal{D} \cup \{A\}$: we show $\mathcal{E}$ satisfies the f.i.c., whence $A \in \mathcal{D}$.

·3· the Tychonoff theorem proven:
let $X = \prod_{\alpha \in J} X_\alpha$, X_α compact, $\mathcal{A}$ be a collection of subsets of X satisfying the f.i.c.: we prove $\bigcap_{A \in \mathcal{A}} \bar{A}$ is nonempty: chose $\mathcal{D} \supset \mathcal{A}$, $\mathcal{D}$ maximal w.r.t. the f.i.c.: then it will suffice to show $\bigcap_{D \in \mathcal{D}} \bar{D}$ is nonempty.
Define $\pi_\alpha : X \to X_\alpha$, the projection map, and consider $\{\pi_\alpha(D) \mid D \in \mathcal{D}\} (\subset X_\alpha)$.
$\forall \alpha$ choose $x_\alpha \in X_\alpha$ s.t. $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$, by compactness. We are done if we show $\vec{x} = (x_\alpha)_{\alpha \in J}$ is in $\bar{D}$ for all $D \in \mathcal{D}$.
Simply enough, if $\pi_\beta^{-1}(U_\beta)$ is any subbasis element containing x, then $\pi_\beta^{-1}(U_\beta)$ intersects every element of $\mathcal{D}$ ($x_\beta \in \overline{\pi_\beta(D)}$, U_β intersects $\pi_\beta(D)$ in some $\pi_\beta(y)$, $y \in D$, so $y \in \pi_\beta^{-1}(U_\beta) \cap D$).
!
Thus every subbasis element containing $\vec{x}$ belongs to $\mathcal{D}$, and thus so does every basis element containing $\vec{x}$ (by ·2·). "Every basis elt. containing $\vec{x}$ intersects every element of $\mathcal{D}$" $\to$ $\vec{x} \in \bar{D}$ $\forall D \in \mathcal{D}$, as desired (!) $\to$ very subtle argument about closures and neighborhoods (!)

▲ **complete regularity** – X is completely regular if 1-point sets are closed in X, and $\forall x_0$, A a closed set not containing x_0, $\exists$ continuous $f : X \to [0,1]$ s.t. $f(x_0) = 1$, $f(A) = 0$.
- normal $\to$ completely regular $\to$ regular
- subspace, product of completely regular is completely regular

● X is completely regular $\to$ X can be imbedded in $[0,1]^J$ for some J.
proof: let $\{f_\alpha\}_{\alpha \in J}$ be the set of all continuous functions $X \to [0,1]$. By complete regularity this collection separates points from closed sets in X. By the imbedding theorem, see Urysohn metrization proof
$F(x) = (f_\alpha(x))_{\alpha \in J}$ is an imbedding of X in $[0,1]^J$.

● X completely regular $\leftrightarrow$ X homeomorphic to a subspace of a compact Hausdorff space

1 point compactification $\leftrightarrow$ X homeomorphic to a subspace of a normal space
"minimal compactification of X"
Stone-Čech $\to$ "maximal compactification of X"

▲ **compactification** of X is a compact Hausdorff space $Y \supset X$ s.t. X is dense in Y. Two compactifications Y_1, Y_2 of X are **equivalent** if $\exists$ homeomorphism $h : Y_1 \to Y_2$ s.t. $\forall x \in X$, $h(x) = x$
- X has a compactification $\leftrightarrow$ X completely regular

@ compactifying a completely regular space: embed $h : X \to Z$ in a compact Hausdorff space Z, and let $X_0 = h(X) \subset Z$, $Y_0 = \bar{X}_0$ in Z. Y_0 is such a compactification. Construct $Y \supset X$ s.t. (X, Y) is homeomorphic to (X_0, Y_0) – just construct it, with a set $A \equiv Y_0 - X_0$, $Y = X \cup A$, in the obvious way.
$\Rightarrow$ compactification of X induced by the imbedding h
$\Rightarrow$ h can be extended to $H : Y \to Z$

question: ? Y a compactification of X, when can a continuous function on X be extended continuously to Y?
- clearly f must be bounded (since its extension, to a compact space, must be so too).

● **Stone-Čech compactification**: let X be completely regular, $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded continuous real-valued functions on X. $\forall \alpha \in J$, choose closed $I_\alpha \in \mathbb{R}$ containing $f_\alpha(X)$ – say, $I_\alpha = [\text{glb } f_\alpha(X), \text{lub } f_\alpha(X)]$, and define $h : X \to \prod_{\alpha \in J} I_\alpha$, by $h(x) = (f_\alpha(x))_{\alpha \in J}$. Now, $\prod_\alpha I_\alpha$ is compact (by Tychonoff) and X is completely regular $\to$ $\{f_\alpha\}$ separates points from closed sets in X $\to$ h is an imbedding, by the imbedding theorem. The compactification of X induced by h is the Stone-Čech compactification of X, $\beta(X)$.

● let X be completely regular, $\beta(X)$ its Stone-Čech compactification; then every bounded continuous real-valued function on X can be uniquely extended to a continuous real-valued function on $\beta(X)$.
proof by the nature of induced compactifications, $\exists$ an imbedding $H : \beta(X) \to \prod I_\alpha$ that equals h when restricted to the subspace X of $\beta(X)$. A continuous bounded real-valued function on X equals f_β, say. If $\pi_\beta : \prod I_\alpha \to I_\beta$ is projection onto the βth coordinate, then $\pi_\beta \circ H : \beta(X) \to I_\beta$ is the desired extension of f_β. To show uniqueness, use the lemma

● $A \subset X$, $f : A \to Z$ continuous, Z Hausdorff: there is at most one extension of f to a continuous $g : \bar{A} \to Z$. *proof*: straightforward proof by contradiction – assume two extensions, and show that if they are diff. at a point of $\bar{A} - A$, they are at a point of A.

idea: the Stone-Čech compactification is unique...

● let X be completely regular, Y_1, Y_2 two compactifications of X having the extension property. $\exists$ homeomorphism ϕ of Y_1 onto Y_2 s.t. $\phi(x) = x$ $\forall x \in X$ – i.e., Y_1, Y_2 are equivalent.
proof: first show that if Y a compactification of X having the extension property, Z a compact Hausdorff space, $g : X \to Z$ a continuous function, g can be extended to a continuous $k : Y \to Z$. Then prove the theorem using inclusion/extension/uniqueness of extension...

(Ch. 6) Metrization & Paracompactness

regular, countable basis $\longrightarrow$ metrizable can we go the other way?

$\hookrightarrow$ can weaken this condition...
basis must be countably locally finite

countable $B = \bigcup_{n \in \mathbb{Z}^+} B_n$, each B_n finite ... $\longrightarrow$ countably locally finite $B = \bigcup_{n \in \mathbb{Z}^+} B_n$, B_n locally finite this gives Nagata–Smirnoff

$\mathcal{D}$ satisfies the f.i.c. if for every finite subcollection of $\mathcal{D}$, their intersection is nonempty.

maximum principle: $\prec$ a strict partial order on A, B a simply ordered subset of A, then there is a maximal simply ordered subset $C \subset A$ containing B.

▲ X a t.s., $\mathcal{A}$ collection of subsets of X is **locally finite** if every point of X has a neighborhood that intersects only finitely many elts. of $\mathcal{A}$.

● $\mathcal{A}$ a locally finite collection of subsets of X:
·1· any subcollection of $\mathcal{A}$ is locally finite.
·2· $\mathcal{B} = \{\bar{A}\}_{A \in \mathcal{A}}$, the collection of the closures of the elts. of $\mathcal{A}$, is l.f.
·3· $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \bar{A}$
proof ·1· obvious. ·2· if open U intersects $\bar{A}$, it intersects A, so U can intersect at most the same number of sets in $\mathcal{B}$. ·3· let $Y = \bigcup_{A \in \mathcal{A}} \bar{A}$: then $\bigcup A \subset Y$, clearly. The reverse assertion requires local finiteness.

▲ $\mathcal{B}$, a collection of subsets of X, is **countably locally finite** if it is the countable union of locally finite collections $\mathcal{B}_n$. also called "σ-locally finite"

We skip (6-2) – (6-5), the Nagata-Smirnoff section p. 247-261,
6-4 is especially useful... return. For now – it's mostly using the same stuff from Urysohn metrization, to formalize

(Ch. 7) Completeness

completeness is a metric property, not a topological one...

▲ **Cauchy sequence** x_n in a metric space (X, d) is s.t. $\forall \epsilon > 0 \exists N$ s.t. $d(x_n, x_m) < \epsilon$ $\forall n, m \ge N$.

▲ **complete** metric space – every Cauchy sequence converges.
- closed subset of complete space is complete
- (X, d) complete, then so is $(X, \bar{d})$ (of the standard bounded metric)
- ● X is complete if every Cauchy sequence in X has a convergent subsequence. *proof*: just show that if the subsequence (x_m) converges so does the original sequence (x_n).

● $\mathbb{R}^n$ is complete in either of its usual metrics, d, ρ. (Euclidean, square)
proof: cook until done.
@ $\mathbb{Q}$, $(-1,1)$, are not complete. $\mathbb{R}^\omega$ is, under the metric which induces the product topology, $D(x,y) = \text{lub}\{\bar{d}(x_i, y_i)/i\}$. $\mathbb{R}^J$ is not complete in general; indeed is not even metrizable if J is uncountable.
BUT if we use the uniform metric $\mathbb{R}^J$ is complete:

▲ **uniform metric** on Y^J: let (Y, d) be a metric space, $\bar{d}(a,b) = \min(d(a,b), 1)$ the standard bounded metric on Y corresponding to d. Given an index set J, define a metric on Y^J, the set of all functions $f : J \to Y$ (!)
$\Rightarrow$ $\bar{\rho}(f,g) = \text{lub}\{\bar{d}(f(\alpha), g(\alpha)) \mid \alpha \in J\}$

theme: functions $\longleftrightarrow$ tuples, functions $\longleftrightarrow$ paths, curves ...

● if Y is complete in the metric d, Y^J is complete in the uniform metric $\bar{\rho}$.
Proof: (Y, d) complete $\Rightarrow$ $(Y, \bar{d})$ complete. Let $f_1, f_2 \dots$ be a Cauchy sequence in Y^J. Then $\forall \alpha \in J$, $f_1(\alpha), f_2(\alpha), \dots$ is Cauchy in $(Y, \bar{d})$ and so converges to, say, y_α by our assumption. Let $f : J \to Y$ $\alpha \mapsto y_\alpha$: then we assert $(f_n) \to f$ in the metric $\bar{\rho}$. The rest of the argument is same as the uniform continuity argument.

Consider Y^X, X a t.s., and consider the subset $\mathcal{C}(X,Y)$ of continuous functions $f : X \to Y$. If Y is complete, so is $\mathcal{C}(X,Y)$!

● let X be a t.s., (Y, d) a metric space. $\mathcal{C}(X,Y)$ is closed in Y^X under the uniform metric. If Y is complete, so is $\mathcal{C}(X,Y)$.
proof: a sequence $f_n \in Y^X$ converges to $f \in Y^X$ rel. to $\bar{\rho}$ implies to f uniformly rel. to $\bar{d}$.
$\Rightarrow$ $\forall \epsilon > 0 \exists N$ s.t. $\bar{\rho}(f, f_n) < \epsilon$, $\forall n \ge N$. Then $\forall x \in X$, $n \ge N$, $\bar{d}(f_n(x), f(x)) \le \bar{\rho}(f_n, f) < \epsilon$, so (f_n) converges uniformly to f.
- $\mathcal{C}(X,Y)$ is closed in Y^X rel. to metric $\bar{\rho}$.
$\Rightarrow$ let $f \in Y^X$ be a limit point of $\mathcal{C}(X,Y)$. Then $\exists$ a sequence $(f_n) \in \mathcal{C}(X,Y)$ converging to f in the metric $\bar{\rho}$. By the uniform limit theorem, f is continuous, so $f \in \mathcal{C}(X,Y)$. Since Y^X is complete, so is the closed subset $\mathcal{C}(X,Y)$.

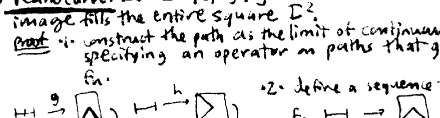
▲ ~~square metric~~ or **sup metric**: X a t.s., (Y, d) a metric space: $\mathcal{F}$ a subset of Y^X s.t. $\forall f, g \in \mathcal{F}$, $\{d(f(x), g(x)) \mid x \in X\}$ is bounded.
sup: supremum Let $\rho(f,g) = \text{lub}\{d(f(x), g(x)) \mid x \in X\}$. This is the **sup metric**.
- $\bar{\rho}(f,g) = \min\{\rho(f,g), 1\}$ relates sup with uniform. – uniform is just the standard bounded metric derived from the sup.
- $\bar{\rho}$, ρ give same topology on $\mathcal{F}$. $\to$ implies equivalent
- $\mathcal{F}$ complete under ρ $\longrightarrow$ $\mathcal{F}$ complete under $\bar{\rho}$

classical theorem ● let (X, d) be a metric space. There is an isometric embedding of X into a complete metric space. *proof*: first we prove a
- **lemma**: let X be a t.s. $\mathcal{B}(X, \mathbb{R})$, the set of all bounded functions $f : X \to \mathbb{R}$, is complete under the sup metric, ρ.
proof: show $\mathcal{B}(X, \mathbb{R})$ is closed in $\mathbb{R}^X$ under $\bar{\rho}$ by considering $(f_n) \to f$ and showing f is bounded.

then consider $\mathcal{B}(X, \mathbb{R})$, $x_0 \in X$, define $\phi_a(x) = d(x,a) - d(x, x_0)$. We assert ϕ_a is bounded. (indeed $|\phi_a(x)| \le d(a, x_0)$.) Define $\Phi : X \to \mathcal{B}(X, \mathbb{R})$ by setting $\Phi(a) = \phi_a$. Show that Φ is an isometric imbedding of X into the complete metric space $\mathcal{B}(X, \mathbb{R})$ – i.e., $\rho(\phi_a, \phi_b) = d(a,b)$. (!)

▲ completion of a metric space X is $h(X)$, where $h : X \to Y$ is an isometric embedding of X into a complete metric space Y.
- uniquely defined up to isometry

● **Peano curve**: Let $I = [0,1]$. $\exists$ a continuous map $f : I \to I^2$ whose image fills the entire square I^2.
proof ·1· construct the path as the limit of continuous functions f_n, by specifying an operator on paths that generates the sequence f_n.
·2· define a sequence $f_n : I \to I^2$



f_2 apply the priming operation to each of the four paths
⋮
f_n 4^n triangular paths, each in a square of size $\frac{1}{2^n}$

- 3 · proof: justifying our choice
  let $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$, $\rho(f,g) = \text{lub}\{d(f(t), g(t)) \mid t \in I\}$. Since I^2 is closed in the square metric, $\mathcal{C}(I, I^2)$ is complete in the metric ρ.
  The sequence (f_n) defined in ·2· is Cauchy: $\rho(f_n, f_{n+m}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m-1}} < \frac{2}{2^n}$ $\forall n, m$.
  since the distance between $f_n(t)$, $f_{n+1}(t)$ is at most $1/2^n$, so $\rho(f_n, f_{n+1}) \leq \frac{1}{2^n}$ (induction)
- 4 · f is surjective is all that remains, now that we've shown f is continuous.
  Let $\vec{x} \in I^2$; let us show $\vec{x} \in f(I)$. Note $\forall n$, f_n comes within $\frac{1}{2^n}$ of x. Clearly $\forall \epsilon$, the ϵ-neighborhood of $\vec{x}$ intersects $f(I)$, simply by choosing f_n, $n > N$, $\frac{1}{2^N} < \frac{\epsilon}{2}$. Thus $\vec{x} \in \overline{f(I)}$, but $f(I)$ is compact (since I is) and so $\vec{x} \in f(I)$, as desired. !

- __compactness__, LPC, and SC (sequential compactness) are the same for __metric spaces__. can we relate __compactness__ and __completeness__?
- every __compact__ space is __complete__. (This is obvious.) Converse not true. Need

▲ __total boundedness__: (X,d) is __totally bounded__ if $\forall \epsilon > 0$, there exists a finite covering of X by ϵ-balls. (total boundedness → boundedness, but not vice versa – one could always pick a bounded metric, e.g.)

● (X,d) is __compact__ iff it's __complete__ and totally bounded. __proof__: compact → complete, totally bounded is obvious. let X be complete and totally bounded: let us prove X is sequentially compact. Let $(x_n) \in X$, and construct a subsequence that is Cauchy (and thus convergent) thus: first cover X with finitely many balls of radius 1. $\forall n$, at least one ball, say B_1, contains ∞ly many

every sequence has a convergent subsequence

- apply result to the function space $\mathcal{C}(X, \mathbb{R}^n)$, X compact.

▲ __equicontinuous__

$\forall f \in \mathcal{F}$

dual to uniformly continuous?

● let X be compact, (Y,d) compact metric. $\mathcal{F}$ equicontinuous $\leftrightarrow$ $\mathcal{F}$ totally bounded in sup metric ρ.

● __classical Ascoli__ X compact, $\mathcal{C}(X, \mathbb{R}^n)$ under the sup metric ρ: $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^n)$ is compact iff it is closed, bounded, and equicontinuous.
  ·1· $\mathcal{F}$ bounded under $\rho \to \exists$ compact $Y \subset \mathbb{R}^n$ s.t. $f(x) \in Y$ $\forall f \in \mathcal{F}$, $\forall x \in X$, so $\mathcal{F} \subset \mathcal{C}(X,Y) \subset \mathcal{C}(X, \mathbb{R}^n)$.

  ·2· $\mathcal{F}$ is equicontinuous.

  ·3· converse

▲ __topology of pointwise convergence__ $x \in X$, open U in Y, let $S(x,U) = \{f \mid f \in Y^X, f(x) \in U\}$ – these sets are a __subbasis__ for the T.O.P.C.
- __basis element__ – finite intersection of subbasis elements $S(x,U)$. (functions that are 'close' to f at finitely many points)
- it is just the __product topology__ !! on Y^J, for $J = X$.

● a sequence $f_n \to f$ in the topology of pointwise convergence __iff__ $\forall x \in X$, the sequence $f_n(x) \to f(x)$.
  __proof__: this is just a fact of the __product topology__.

▲ __topology of compact convergence__ (Y,d) a metric space, X a t.s., $f \in Y^X$; compact $C \subset X$, $\epsilon > 0$. Let $B_C(f,\epsilon)$ = the set of $g \in Y^X$ such that $\text{lub}\{d(f(x), g(x)) \mid x \in C\} < \epsilon$ (i.e., those which fit in a narrow space about a function $f(x)$)
– these sets form a basis for the T.O.C.C.
- if $g \in B_C(f,\epsilon)$ $\exists \delta > 0$ s.t. $B_C(g,\delta) \subset B_C(f,\epsilon)$ – just let $\delta = \epsilon - \text{lub}\{d(f(x), g(x)) \mid x \in C\}$

● a sequence $f_n : X \to Y$ of functions converges to f in the T.O.C.C.